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New Procedure for Deriving Optimized Strapdown Attitude Algorithms

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A new procedure for deriving strapdown attitude algorithms is described and justified. This procedure allows optimization of the solution when the vehicle angular rate components are known analytically. It is based on Miller's approach, but it uses the analytical relationship between angular rate derivatives and does not require the derivation of an analytical expression for the error quaternion. The procedure was tested for both classical and more general conic motion, when vehicle angular rate components are described as the Jacobian elliptic functions. It is shown that the coefficients optimized for classical coning hold true for general coning as well but only if the rate component peak values are properly specified and the third term of the rotation vector differential equation is taken into account. A new kinematically correct description for the generalized coning is presented, and two variants of the angular rate's representation, which meet this criterion, are derived. Finally, a statistical refinement of the deterministic Miller's procedure (Miller, R. B., "A New Strapdown Attitude Algorithm," *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 4, 1983, pp. 287–291), which imparts smoothing properties to an algorithm, is formulated, and an example of its application is presented.

I. Introduction

AMONG the multitude of papers devoted to the development of the strapdown attitude algorithms, Miller's is undoubtedly a fundamental one.[1] A general procedure for deriving the attitude algorithm is presented in it. The procedure is broken down into two steps that are considered independently: 1) the calculation of the rotation vector, which determines the vehicle attitude change over the iteration interval, and 2) the quaternion update to determine the current vehicle's attitude relative to some reference frame. The form of the rotation vector algorithm is uniquely determined by the chosen type of the angular rate's polynomial model, which depends on how many gyro samples over the iteration interval are available. Miller has also suggested the procedure for optimizing the derived attitude algorithm for the pure conic motion, which is traditionally considered as a test input to evaluate the algorithm's performance. The procedure is based on deriving the analytical expression for the corrective quaternion generated by the algorithm and allows improvement of the algorithm's accuracy by the appropriate setting of its coefficients.

Miller's procedure[1] for optimizing the rotation vector algorithm has been applied to the different algorithms in Refs. 2–4. In all of these papers the idea of Miller's procedure has remained unchanged and the classical coning has been used as test input for the algorithms.

The tuning of an algorithm to classical coning allows higher computational accuracy in comparison with the general case of motion without adding complexity to the algorithm. Moreover, if the conic frequency is known exactly, the computational drift can be fully compensated even with the simplest attitude algorithm.[2] However, one can legitimately doubt that the real vehicle's motion is a pure coning. Hence it is not correct to use the algorithm, which is closely tuned on the specific type of motion, even if this motion is the worst case from the viewpoint of the computational errors (as which the pure coning).

The investigation of the algorithm's performance in more general environments typically experienced in real applications was presented in Refs. 4 and 5. In Ref. 4, a case was examined where body angular rate components are represented as the harmonic series decomposition. It was shown that the algorithms optimized for the classical coning also maximize their performance in the generalized oscillatory environment. An alternative description for the angular motion was suggested in Ref. 5. It was justified that the generalized conic motion, when the angular rate components are the Jacobian elliptic functions, can be used as the test input to the algorithm. It is reasonable to optimize the algorithms for such type of motion. However, the use of Miller's optimizing procedure would be prohibitive in this case because the analytical expression for the quaternion true value is unknown. Also the simulation results presented by the authors have shown that the optimization as based on classical coning does not work for the Jacobian elliptic functions.

In the present paper, a new procedure for optimizing the attitude algorithms based on Miller's general approach is derived. With known analytical relationships between the angular rate derivatives, the procedure allows optimization of the algorithm without the use of the expressions for both the true quaternion value and the one generated by the algorithm. Owing to this feature, the suggested procedure is more universal than Miller's and can be applied to generalized coning as it is specified in Ref. 5.

The short description of Miller's general approach as a starting point for this study comes before the presentation of the new results. The procedure itself is applied to the case of three gyro samples over the iteration interval to have an opportunity to compare the results with the known ones. The procedure is tested on both the classical and the generalized conic motion, and in the process, the analytical results are compared with the simulation results. In the process of this research, the importance of the contribution of the third term in the rotation vector differential equation (which conventionally is discarded) and the importance of correctly setting the angular rate's peak values for the generalized coning are revealed and studied. The general kinematically correct description for the generalized conic motion is presented. Two alternative representations for angular rate

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components that meet this approach are derived. In the concluding section of the paper, the statistical refinement of Miller's initial procedure (which is deterministic) is stated. According to this approach, measurement noise smoothing can be inserted into the procedure and the algorithm can be optimized from this point of view. An example of an algorithm based on the least-squares method with Chebyshev orthogonal polynomials is presented to illustrate the procedure.

II. Miller's Procedure

According to Ref. 1, the strapdown attitude algorithm is performed in two steps: 1) calculation of the rotation vector, which characterizes the vehicle's attitude variation during the iteration interval h, and 2) update of the quaternion, which characterizes the vehicle's current orientation relative to some reference frame.

A general algorithm for updating the quaternion $\boldsymbol{Q}$ is

$$\boldsymbol{Q}(T+h) = \boldsymbol{Q}(T) * \boldsymbol{q}(h) \tag{1}$$

where $\boldsymbol{Q}(T)$ and $\boldsymbol{Q}(T+h)$ are the previous and the current quaternion values, respectively, and $q(h)$ is the updating quaternion, which is related to the rotation vector $\boldsymbol{\Phi}$ as follows:

$$\boldsymbol{q}(h) = \begin{bmatrix} C \\ S \cdot \Phi_x \\ S \cdot \Phi_y \\ S \cdot \Phi_z \end{bmatrix} \tag{2}$$

where $C = \cos(\Phi_0/2)$, $S = (1/\Phi_0)\sin(\Phi_0/2)$, $\Phi_0 = (\boldsymbol{\Phi}\cdot\boldsymbol{\Phi})^{1/2}$ is the rotation vector magnitude, and Φ_x, Φ_y, and Φ_z are the rotation vector body frame components. The rotation vector differential equation can be written as

$$\dot{\boldsymbol{\Phi}} = \omega + \tfrac{1}{2}(\boldsymbol{\Phi}\times\omega) + A\boldsymbol{\Phi}\times(\boldsymbol{\Phi}\times\omega) \tag{3}$$

where ω is the vehicle angular rate vector and

$$A = \frac{1}{\Phi_0^2}\left[1 - \frac{\Phi_0\sin\Phi_0}{2(1-\cos\Phi_0)}\right] \tag{4}$$

When Φ_0 is small enough, A can be approximated by $A \approx \frac{1}{12}$. Equation (3) is conventionally simplified by discarding the third term and substituting for $\boldsymbol{\Phi}$ with its first-order solution in the second term, i.e.,

$$\dot{\boldsymbol{\Phi}} = \omega + \tfrac{1}{2}(\boldsymbol{\Phi}_1\times\omega) \tag{5}$$

$$\dot{\boldsymbol{\Phi}}_1 = \boldsymbol{\omega} \tag{6}$$

The solution of Eq. (5) is sought as a Taylor series expansion

$$\boldsymbol{\Phi}(T+h) = \boldsymbol{\Phi}(T) + \dot{\boldsymbol{\Phi}}(T)h + \ddot{\boldsymbol{\Phi}}(T)(h^2/2) + \boldsymbol{\Phi}^{(3)}(h^3/6) + \cdots \tag{7}$$

Repeatedly differentiating Eq. (5) in view of Eq. (6), one can obtain the relationship between the rotation vector derivatives at the time point $t = T$ and the angular rate derivatives at the same point. Therefore, for the square-law mode for vector $\boldsymbol{\omega}(t)$ on the time interval $(T, T+h)$, we have

$$\boldsymbol{\omega}(T+\upsilon) = \boldsymbol{a} + 2\boldsymbol{b}\upsilon + 6\boldsymbol{c}(\upsilon^2/2) \tag{8}$$

where

$$\begin{gathered} \boldsymbol{a} = \boldsymbol{\omega}(T), \qquad 2\boldsymbol{b} = \dot{\boldsymbol{\omega}}(T), \qquad 6\boldsymbol{c} = \ddot{\boldsymbol{\omega}}(T) \\ \boldsymbol{\omega}^{(i)}(T) = 0 \qquad \text{for} \qquad i \geq 3 \end{gathered} \tag{9}$$

and taking into account that $\boldsymbol{\Phi}(T) = 0$, there results

$$\begin{gathered} \dot{\boldsymbol{\Phi}}(T) = \boldsymbol{a}, \qquad \ddot{\boldsymbol{\Phi}}(T) = 2\boldsymbol{b}, \qquad \boldsymbol{\Phi}^{(3)}(T) = 6\boldsymbol{c} + (\boldsymbol{a}\times\boldsymbol{b}) \\ \boldsymbol{\Phi}^{(4)}(T) = 6(\boldsymbol{a}\times\boldsymbol{c}), \qquad \boldsymbol{\Phi}^{(5)}(T) = 6(\boldsymbol{b}\times\boldsymbol{c}) \\ \boldsymbol{\Phi}^{(j)}(T) = 0 \qquad \text{for} \qquad j \geq 6 \end{gathered} \tag{10}$$

Substituting Eq. (10) into Eq. (7) yields

$$\begin{aligned} \boldsymbol{\Phi}(T+h) &= \boldsymbol{a}h + \boldsymbol{b}h^2 + \boldsymbol{c}h^3 + (h^3/6)(\boldsymbol{a}\times\boldsymbol{b}) \\ &\quad + (h^4/4)(\boldsymbol{a}\times\boldsymbol{c}) + (h^5/10)(\boldsymbol{b}\times\boldsymbol{c}) \end{aligned} \tag{11}$$

The coefficients $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ can be determined from gyro outputs. In fact, as the gyro output's vector by definition is

$$\boldsymbol{\Theta}(\upsilon_1, \upsilon_2) = \int_{\upsilon_1}^{\upsilon_2} \boldsymbol{\omega}(T+\upsilon)\,\mathrm{d}\upsilon \tag{12}$$

then in view of Eq. (8), it gives

$$\boldsymbol{\Theta}(\upsilon_1, \upsilon_2) = \boldsymbol{a}(\upsilon_2 - \upsilon_1) + \boldsymbol{b}\left(\upsilon_2^2 - \upsilon_1^2\right) + \boldsymbol{c}\left(\upsilon_2^3 - \upsilon_1^3\right) \tag{13}$$

Consequently, at least three gyro samples over the iteration interval must be available to determine three coefficients $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. If we have the gyro outputs at time points $t = T + h/3$, $t = T + 2h/3$, and $t = T + h$, which we denote by $\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_2$, and $\boldsymbol{\Theta}_3$, respectively, the set of three linear inhomogeneous algebraic equations can be derived. Its solution can be written as

$$\begin{aligned} \hat{\boldsymbol{a}} &= (1/2h)(11\boldsymbol{\Theta}_1 - 7\boldsymbol{\Theta}_2 + 2\boldsymbol{\Theta}_3) \\ \hat{\boldsymbol{b}} &= (9/2h^2)(-2\boldsymbol{\Theta}_1 + 3\boldsymbol{\Theta}_2 - \boldsymbol{\Theta}_3) \\ \hat{\boldsymbol{c}} &= (9/2h^3)(\boldsymbol{\Theta}_1 - 2\boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3) \end{aligned} \tag{14}$$

where $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{c}}$ are the appropriate coefficient's estimates.

Substituting these estimates into Eq. (11) and taking into account Eq. (13) result in

$$\boldsymbol{\Phi}(T+h) = \boldsymbol{\Theta} + X(\boldsymbol{\Theta}_1\times\boldsymbol{\Theta}_3) + Y\boldsymbol{\Theta}_2\times(\boldsymbol{\Theta}_3 - \boldsymbol{\Theta}_1) \tag{15}$$

where

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3, \qquad X = 0.4125, \qquad Y = 0.7215 \tag{16}$$

Miller suggested the procedure for optimizing the algorithm's coefficients X and Y for classical coning. There is no need to present this procedure in detail here. But its key feature must be underlined: the procedure is based on the derivation of analytical expressions for the rotation vector estimate and (with the known expression for the coning rotation vector true value) error quaternion. The criterion for the algorithm's optimization is the minimization of the zero frequency component of updating quaternion error. The numerical values of the optimized coefficients are

$$X = 0.45, \qquad Y = 0.675 \tag{17}$$

A modification of Miller's procedure was suggested in Ref. 4. It is more convenient than the original one because it uses the analytical expressions for the coning correction term (instead of the quaternion), which have a more compact form for the coning input.

Thus the known procedures for optimizing the algorithm do not depend on the procedure for deriving the algorithm but only presuppose that a form of the algorithm is given.

III. New Procedure

The procedure to be presented here deals with the approach to the algorithm's optimization for the case of analytically defined test motion. It is directly based on the procedure for deriving the algorithm, and thus the optimization is achieved by going from a general procedure to a special one when analyzing the algorithm's error. In doing so, the polynomial description of motion is used, which is known to be adequate for any differentiable functions.[6] By practice, time intervals are chosen to be small enough to allow piecewise polynomial approximations to harmonic motion inputs using a limited number of coefficients in the polynomial. The main idea of this procedure is 1) to obtain the expression for the alogirthm's computational error in the general form as a function of an additional angular rate model's coefficient and 2) to drive a relationship between this unknown coefficient and the known ones to make the error compensation possible.

Let us derive the refined expression for the rotation vector with the availability of three gyro samples over the iteration interval but for the more exact approximation of the angular rate vector. Let $\boldsymbol{\omega}(t)$ be described by a cubic [not quadratic as in Eq. (8)] polynomial on the interval $(T, T+h)$, namely,

$$\boldsymbol{\omega}(T+\upsilon) = \boldsymbol{a} + 2\boldsymbol{b}\upsilon + 6\boldsymbol{c}(\upsilon^2/2) + 24\boldsymbol{d}(\upsilon^3/6) \tag{18}$$

where

$$24\boldsymbol{d} = \boldsymbol{\omega}^{(3)}(T) \tag{19}$$

The expressions (10) for rotation vector derivatives in the time point $t = T$ become

$$\dot{\boldsymbol{\Phi}}(T) = \boldsymbol{a}, \qquad \ddot{\boldsymbol{\Phi}}(T) = 2\boldsymbol{b}$$

$$\boldsymbol{\Phi}^{(3)}(T) = 6\boldsymbol{c} + (\boldsymbol{a}\times\boldsymbol{b}), \qquad \boldsymbol{\Phi}^{(4)}(T) = 24\boldsymbol{d} + 6(\boldsymbol{a}\times\boldsymbol{c})$$

$$\boldsymbol{\Phi}^{(5)}(T) = 12(\boldsymbol{b}\times\boldsymbol{c}) + 36(\boldsymbol{a}\times\boldsymbol{d}) \tag{20}$$

$$\boldsymbol{\Phi}^{(6)}(T) = 120(\boldsymbol{b}\times\boldsymbol{d}), \qquad \boldsymbol{\Phi}^{(7)}(T) = 360(\boldsymbol{c}\times\boldsymbol{d})$$

$$\boldsymbol{\Phi}^{(j)}(T) = 0 \qquad \text{for} \qquad j \geq 8$$

Now let us evaluate the impact of the cubic term in the model (18) on estimates of coefficients $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, which are calculated according to Eq. (14). Passing on from the model (13) to the refined model for Θ,

$$\Theta(\upsilon_1, \upsilon_2) = \boldsymbol{a}(\upsilon_2 - \upsilon_1) + \boldsymbol{b}\left(\upsilon_2^2 - \upsilon_1^2\right) + \boldsymbol{c}\left(\upsilon_2^3 - \upsilon_1^3\right) + \boldsymbol{d}\left(\upsilon_2^4 - \upsilon_1^4\right) \tag{21}$$

For $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, it can be found that

$$\boldsymbol{a} = \hat{\boldsymbol{a}} - (2h^3/9)\boldsymbol{d}, \qquad \boldsymbol{b} = \hat{\boldsymbol{b}} - (11h^2/9)\boldsymbol{d}$$
$$\boldsymbol{c} = \hat{\boldsymbol{c}} - (18h/9)\boldsymbol{d} \tag{22}$$

where the parameters $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{c}}$, are calculated via gyro outputs according to the expressions (14).

Substituting Eq. (22) into Eq. (20), and then Eq. (20) into Eq. (7), finally results in

$$\boldsymbol{\Phi}(T+h) = \boldsymbol{a}h + \boldsymbol{b}h^2 + \boldsymbol{c}h^3 + \boldsymbol{d}h^4 + (h^3/6)(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})$$
$$+ (h^4/4)(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{c}}) + (h^5/10)(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{c}}) + (h^5/270)(\boldsymbol{a}\times\boldsymbol{d})$$
$$+ (h^6/270)(\boldsymbol{b}\times\boldsymbol{d}) + (h^7/210)(\boldsymbol{c}\times\boldsymbol{d}) \tag{23}$$

Comparing Eq. (23) with Eq. (11), one can obtain the expression for the rotation vector calculation error $\Delta\,\boldsymbol{\Phi}$ for the case when algorithms (15) and (16) are used. If only the most significant term is taken into account, this expression can be written as

$$\boldsymbol{\Phi}(T+h) = (h^5/270)(\boldsymbol{a}\times\boldsymbol{d}) \tag{24}$$

The coefficient $\boldsymbol{d}$ is unknown in the general case, and so the idea of the algorithm's optimization is to express $\Delta\,\boldsymbol{\Phi}$ via the known estimates $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{c}}$ for the analytically specified vehicle's motion and to compensate it in the algorithm.

To test the suggested procedure, we will try to optimize Miller's algorithm for classical coning.

The angular rate vector for classical coning is[1]

$$\boldsymbol{\omega} = \begin{bmatrix} -2\omega_0\sin^2(a/2) \\ -\omega_0\sin(a)\sin(\omega_0 t) \\ -\omega_0\sin(a)\cos(\omega_0 t) \end{bmatrix} \tag{25}$$

where a is a coning half-angle and ω_0 is a coning circular frequency.

It is well known for classical coning that the computational drift appears only in the second element of the quaternion and is caused by the nonzero average error in rotation vector along the coning axis (x). Consequently, as a criterion for the algorithm's optimization, we will select the minimization of the $\Delta\,\boldsymbol{\Phi}_x$ component.[2]

Differentiating Eq. (25) and accounting for the relationship between angular rate derivations at the time point $t = T$ and the coefficients $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ [see expressions (9) and (19)], one can show that, for classical coning,

$$(\boldsymbol{a}\times\boldsymbol{d})_x = -\tfrac{1}{2}(\boldsymbol{b}\times\boldsymbol{c})_x \tag{26}$$

In view of Eq. (22), the expression (26) becomes

$$(\boldsymbol{a}\times\boldsymbol{d})_x = -\tfrac{1}{2}(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{c}})_x + h(\boldsymbol{b}\times\boldsymbol{d})_x + \tfrac{11}{18}h^2(\boldsymbol{c}\times\boldsymbol{d})_x \tag{27}$$

And so it becomes possible to calculate the rotation vector error, which is defined by Eq. (24), via the known parameters $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{c}}$ and to compensate it in the algorithm.

The refined algorithm for the rotation vector calculation is

$$\boldsymbol{\Phi}(T+h) = \boldsymbol{a}h + \boldsymbol{b}h^2 + \boldsymbol{c}h^3 + \boldsymbol{d}h^4 + (h^3/6)(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})$$
$$+ (h^4/4)(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{c}}) + h^5\tfrac{53}{540}(\hat{\boldsymbol{b}}\times\hat{\boldsymbol{c}}) \tag{28}$$

Substituting the expressions (14) for $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{c}}$ into Eq. (28) and taking into account the relation (21), we finally obtain the algorithm for $\boldsymbol{\Phi}$ in the form of Eq. (15) with the coefficients

$$X = \tfrac{9}{20}, \qquad Y = \tfrac{27}{40} \tag{29}$$

It is easy to see that the derived values are the same as Miller's optimized coefficients in Eq. (17). Taking into account that, for classical coning,

$$\boldsymbol{b}\times\boldsymbol{d} = 0 \tag{30}$$

the x axis component of the rotation vector's error can be written as

$$\Delta\,\boldsymbol{\Phi}_x = \frac{a^2h^7\omega_0^7}{204{,}120} \tag{31}$$

Note that for Miller's algorithm with unoptimized coefficients the rotation vector's error, which is defined by the expression (24), can be written as

$$\Delta\,\boldsymbol{\Phi}_x = \frac{a^2h^5\omega_0^5}{6480} \tag{32}$$

The expressions (31) and (32) are the same as presented by previous authors.[1,2,5]

The perfect agreement of the derived values of the optimized algorithm's coefficients and the analytical expressions for the rotation vector's error for both optimized and unoptimized Miller's algorithms under classical coning input with those determined earlier proves the correctness of the suggested procedure.

Thus, the preceding procedure allows us to optimize Miller's algorithms for classical coning but in distinction to Miller's procedure without using the analytical expressions for true and estimated values of the attitude parameters. In the next section, it will be applied to generalized conic motion.

IV. Generalized Conic Motion

As noted earlier, the exact tuning of the attitude algorithm on the specific test input proves out only when this model is adequate. Classical coning seems to be a too idealized model for the single reason that the precession's angular rate relative to the x axis is considered to be constant.

A generalized conic motion was suggested for use as an algorithm's test input.[5] Under this motion the angular rate's components are described by Jacobian elliptic functions as follows:

$$\boldsymbol{\omega} = \begin{bmatrix} a_y\cdot\mathrm{cn}(u,m) \\ a_y\cdot\mathrm{sn}(u,m) \\ a_z\cdot\mathrm{dn}(u,m) \end{bmatrix} \tag{33}$$

when dn, sn, and cn are the Jacobian elliptic functions[7,8]; m is a Jacobian function's parameter; a_x, a_y, and a_z are the half-angles along the appropriate axes; and $u = \omega_0 t$.

The authors justify the use of the Jacobian elliptic functions with three reasons.

1) From the formally mathematical point of view, a more general description must involve the classical coning as a special case. And this is so. Indeed, as m approaches zero, cn becomes cos, sn becomes sin, and dn becomes a unit constant.

2) From the physical point of view, the solution of Euler's equations of motion for an asymmetric, torque-free rigid body can be expressed in terms of Jacobian elliptic functions (a mathematical pendulum is one such example).

3) From the viewpoint of the algorithm's computational drift, the classical coning is the worst case only when the algorithm is used in the form of Eqs. (3) and (4). When the algorithm is used in the form of Eqs. (5) and (6) (which is indeed the case), the noncommutativity rate vector has a maximum under the condition

$$\boldsymbol{\omega} \perp \boldsymbol{\Phi}_1 \tag{34}$$

This condition is met with the generalized (not classical) coning motion when the angular rate's components are the Jacobian elliptic functions.

While basically agreeing with these arguments, we are of the opinion that the expressions (33) for vector $\boldsymbol{\omega}$ components must be refined: it is necessary 1) to introduce ω_0 as a factor to all of the angular rate's components to give them a proper dimensional representation and 2) to satisfy the signs of components and the relations between their peak values, which are peculiar to conic motion. In view of remarks we have made, the expressions for ω become

$$\boldsymbol{\omega} = \begin{bmatrix} -\left(a_x^2/2\right)\omega_0 \mathrm{dn}(u,m) \\ -a_y\omega_0 \mathrm{sn}(u,m) \\ -a_z\omega_0 \mathrm{cn}(u,m) \end{bmatrix} \tag{35}$$

In expressions (35), the components along the x and z axes are interchanged to agree with expressions (25) for classical coning. In our opinion, some other coning representations for vector ω under generalized conic motion are acceptable as well. Two of them will be discussed later.

Now we will attempt to optimize Miller's algorithm for generalized coning in the form (35) with the procedure presented earlier. Assuming that in this case, too, the instantaneous rotation axis is orthogonal to an x axis, i.e., $\boldsymbol{\Phi}_x = 0$, the criterion for optimization we will take is the same as for classical coning. Let us evaluate whether the condition of Eq. (26) for optimizing the algorithm under the classical coning holds true for the generalized coning as well. For this purpose, let us represent the Jacobian elliptic functions as a trigonometrical series[7]:

$$\begin{aligned} \mathrm{dn}(u,m) &= \frac{\pi}{2K(m)} \times \left[1 + \frac{2\cos(2u_1)}{ch(2\rho)} + \frac{2\cos(4u_1)}{ch(4\rho)} + \frac{2\cos(6u_1)}{ch(6\rho)} + \cdots\right] \\ \mathrm{sn}(u,m) &= \frac{\pi}{\sqrt{m}K(m)} \times \left[1 + \frac{\sin(u_1)}{sh(\rho)} + \frac{\sin(3u_1)}{sh(3\rho)} + \frac{\cos(5u_1)}{sh(5\rho)} + \cdots\right] \\ \mathrm{cn}(u,m) &= \frac{\pi}{\sqrt{m}K(m)} \times \left[1 + \frac{\cos(u_1)}{ch(\rho)} + \frac{\cos(3u_1)}{ch(3\rho)} + \frac{\cos(5u_1)}{ch(5\rho)} + \cdots\right] \end{aligned} \tag{36}$$

where $u_1 = \omega_1 t$, $\omega_1 = [\pi/22K(m)]\omega_0$, $\rho = (\pi/2)\cdot[K(1-m)/K(m)]$, $K(m)$, and $K(1-m)$ are the ***I***st genus total elliptic integral's values for the basic (m) and the additional $(1-m)$ Jacobian function's parameters, respectively. Using the representation of Eq. (36), it is not difficult to derive the expressions for the angular rate's derivatives, which will be written as a trigonometrical series as well. With the known relationship between the angular rate's derivatives and the angular rate model's parameters $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ [see relations (9) and (19)], it can be shown that in vector products $\boldsymbol{a}\times\boldsymbol{d}$ and $\boldsymbol{b}\times\boldsymbol{c}$ the zero frequency terms are separated out only in x-axis components, and the relation (26) holds true. The concrete expression for the nonzero average x-axis component of the vector products under consideration is

$$\overline{(\boldsymbol{a}\times\boldsymbol{d})}_x = -\frac{1}{2}\overline{(\boldsymbol{b}\times\boldsymbol{c})}_x = a_y a_z \omega_0^5 \cdot \frac{\pi^5}{8mK^5(m)} \times \left[\frac{1}{sh(\rho)ch(\rho)} + \frac{3^3}{sh(3\rho)ch(3\rho)} + \frac{5^3}{sh(5\rho)ch(5\rho)} + \cdots\right] \tag{37}$$

Thus with the suggested procedure, we managed to prove analytically that the conditions for optimizing Miller's algorithm are the same for both classical and generalized (with arbitrary m from 0 to 1) conic motion; they are provided by the algorithm of Eq. (15) with the coefficients of Eq. (17). To test the validity of this conclusion, a numerical simulation was held. The simulation strategy and the obtained results will be discussed next.

V. Simulation

The procedure for computing the attitude algorithm's performance by means of the simulation was taken from Ref. 5. The substance of this procedure is as follows (in short).

The computational drift in the error quaternion components is taken to be a measure of the algorithm's performance. The error quaternion for the algorithm with a quaternion updating rate of f_1 is calculated as

$$\delta\boldsymbol{Q}_{f_1} = \boldsymbol{Q}_{f_0} * \boldsymbol{Q}^*_{f_1} \tag{38}$$

where $\boldsymbol{Q}_{f_0}$ and $\boldsymbol{Q}^*_{f_1}$ are the quaternion and cojugate quaternion, respectively, which are calculated with the same algorithm under investigation but with a different updating rate f_0 and $f_1 : f_0 \gg f_1$. Therefore, the quaternion $\boldsymbol{Q}_{f_0}$ is used as an estimate of the true quaternion value. Note that this procedure is appropriate for use even in those cases when the exact analytical expression for the quaternion true value is known (as for the classical coning) because the errors of high-frequency primary integration for the gyro samples simulation have no influence on the $\delta\boldsymbol{Q}$ calculation accuracy.

The computational drift $\delta\dot{\boldsymbol{Q}}_{f_i}$ can be represented as

$$\delta\dot{\boldsymbol{Q}}_{f_i} = K(h_i)^r \tag{39}$$

where $h_i = 1/f_i$ is an iteration (updating) interval, K is a constant coefficient for the algorithm being investigated, and r is an iteration interval exponent that characterizes the order of an algorithm's accuracy.

To evaluate the algorithm's parameters K and r, the computational drift $\delta\dot{\boldsymbol{Q}}_{f_i}$ is determined for two different iteration rate's values, f_1 and f_2. Taking into account that the computational drift in the quaternion $\boldsymbol{Q}_{f_0}$ is rather small (relative to the quaternions $\boldsymbol{Q}_{f_1}$ and $\boldsymbol{Q}_{f_2}$) for the parameter K and r estimates, it can be approximately written

$$r = \frac{\ln\left[\delta Q_{f_{1x}}/\delta Q_{f_{2x}}\right]}{\ln[h_1/h_2]}, \qquad K = \frac{\delta Q_{f_{1x}}}{(h_1)^r} \tag{40}$$

The simulation program's debugging was performed on the classical coning with the following iteration rate's values:

$$f_0 = 30{,}000\ \mathrm{Hz}, \qquad f_1 = 300\ \mathrm{Hz}, \qquad f_2 = 150\ \mathrm{Hz} \tag{41}$$

The time interval for computational drift evaluation was equal to 1 s. The angular rates were simulated in the form of Eq. (25) with parameters

$$\omega_0 = 2\pi \times 10\ \mathrm{s}^{-1}, \qquad a = 1\ \mathrm{deg} \tag{42}$$

The quaternion initial value was specified as

$$\boldsymbol{Q}_{f_0}(0) = \boldsymbol{Q}_{f_1}(0) = \begin{bmatrix} \cos(a/2) \\ 0 \\ \sin(a/2) \\ 0 \end{bmatrix} \tag{43}$$

The simulation results for the classical coning agreed closely with analytical estimates and those obtained in Ref. 5. This fact demonstrates the correctness of the simulation program.

Table 1 shows the simulation results for the generalized coning with the angular rate in the form of Eq. (35) and for four different parameter m values. Three harmonics were retained from the infinite series for each Jacobian elliptic function in Eq. (35). The iteration rates for the simulation were taken to be the same as those for the

Table 1 Performance of the truncated algorithm

m	Axis	$\delta\dot{Q}_{150}$ Nonoptimized	$\delta\dot{Q}_{150}$ Optimized	$\delta\dot{Q}_{300}$ Nonoptimized	$\delta\dot{Q}_{300}$ Optimized	r Nonoptimized	r Optimized
0.1	x	-4.15×10^{-8}	2.02×10^{-10}	-2.65×10^{-9}	2.8×10^{-12}	3.97	6.15
	z	-7.4×10^{-10}	——	-4.7×10^{-11}	——	3.98	——
0.3	x	-3.5×10^{-8}	0.5×10^{-15}	-2.2×10^{-9}	-0.2×10^{-12}	3.99	——
	z	-6.4×10^{-10}	——	-4.1×10^{-11}	——	3.96	——
0.5	x	-2.9×10^{-8}	0.5×10^{-15}	-1.8×10^{-9}	0.2×10^{-11}	3.98	5.45
	z	-5.6×10^{-10}	——	-3.8×10^{-11}	——	3.96	——
0.7	x	-2.2×10^{-8}	-0.52×10^{-9}	-1.4×10^{-9}	-0.4×10^{-11}	3.97	3.78
	z	-5.0×10^{-10}	——	-3.3×10^{-11}	——	3.92	——

classical coning, Eq. (41). The numerical values for the parameters were as follows:

$$\omega_0 = 2\Pi \times 10\ \mathrm{s}^{-1}, \qquad a_x = a_y = a_z = 1\ \mathrm{deg}$$
$$m = 0.1;\ 0.3;\ 0.5;\ 0.7 \tag{44}$$

The elliptic integral's values $K(m)$ and $K(1-m)$ were taken from the appropriate tables.[8]

As Table 1 illustrates, for small m ($m = 0.1$), the algorithm approaches an optimized one ($r \approx 6$), but as m increases, the algorithm loses this property (r approaches 4). Note that all of this occurs even with equal values of a_x, a_y, and a_z. Based on these results, it is safe to assume that in parallel with the computational error due to the discrepancy of the angular rate's model over the iteration interval (which minimization was the subject of the procedure described earlier) there exists some additional error of another character that varies as the fifth power of h (for $\Delta\,\mathbf{\Phi}$). In other words, for Miller's algorithm, $\Delta\,\mathbf{\Phi}$ can be represented as a sum

$$\Delta\,\mathbf{\Phi} = \Delta\,\mathbf{\Phi}_1 + \Delta\,\mathbf{\Phi}_2 \tag{45}$$

where $\Delta\,\mathbf{\Phi}_1 = Kh^7$ and $\Delta\,\mathbf{\Phi}_2 = Mh^5$. It is evident that if $\Delta\,\mathbf{\Phi}_1 \gg \Delta\,\mathbf{\Phi}_2$, an exponent r approaches 6, and if $\Delta\,\mathbf{\Phi}_2 \gg \Delta\,\mathbf{\Phi}_1$, r approaches 4. In those cases when $\Delta\,\mathbf{\Phi}_1$ and $\Delta\,\mathbf{\Phi}_2$ are of the same order, the estimate of r will be somewhat in the range $4 < r < 6$ if K and M are of the same signs, and it does not exist in principle if K and M are of the opposite sign. Turning back to Table 1, it would be logical to suppose that for small $m \Delta\,\mathbf{\Phi}_1$ prevails and for large $m \Delta\,\mathbf{\Phi}_2$ prevails, and K and M are of the opposite signs.

Note in Table 1 that the performance of the nonoptimized algorithm remains virtually the same for all values of m (including the largest value of 0.7). This indicates that there are no deleterious effects due to the change in the harmonic contents of the motion at the higher values of m. It is the possibility of further improvement by optimization that appears to decline as m increases. A numerical evaluation of the elliptic functions sn, cn, and dn also shows negligible harmonics higher than about 20 Hz for sn and cn and about 30 Hz for dn for the case of $m = 0.7$. On the other hand there are no low-frequency harmonics as well, because a fundamental frequency [from Eq. (36)] is about 7.5 Hz. Hence, a 1-s simulation interval is suitable for $m = 0.7$.

We suppose that there is no source of such an additional error except the discarding of the third term in the rotation vector differential equations. The investigation of this problem is the goal of the next section.

VI. Contribution of the Third Term of Eq. (3)

Let us derive, using Miller's procedure, the algorithm for the rotation vector based on the original differential equation in the form of Eq. (3) to evaluate the contribution of the third term of Eq. (3). It can be shown that the expressions for vector $\mathbf{\Phi}$ derivations will be refined from the fifth derivative and on. Restricting our consideration to the fifth derivative, which has a dominant effect for the additional term in $\mathbf{\Phi}^{(5)}(T)$, it can be written (for small $\mathbf{\Phi}_0$, when $A \approx \frac{1}{12}$)

$$\Delta\,\mathbf{\Phi}^{(5)}(T) = \boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{c}) + 2\boldsymbol{b}\times(\boldsymbol{b}\times\boldsymbol{a}) + \tfrac{1}{3}\boldsymbol{a}\times[(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}] \tag{46}$$

The expression (46) determines the error in calculating $\mathbf{\Phi}^{(5)}(T)$, which is introduced when Miller's algorithms (15) with both optimized and nonoptimized coefficients are used. Going to the rotation vector's error according to Eq. (7), we obtain

$$\Delta\,\mathbf{\Phi}_2 = \Delta\,\mathbf{\Phi}^{(5)}(T)(h^5/120) \tag{47}$$

And so the error due to the discarding of the third term in Eq. (3) varies as a fifth power of h, and hence it can act as $\mathbf{\Phi}_2$ error in Eq. (45). To confirm this assumption, the simulation of the exact Miller algorithm (with the third term involved) with optimized coefficients was performed under generalized coning. The rotation vector algorithm was realized in the following form:

$$\mathbf{\Phi}(T+h) = K_1\Theta_1 + K_2\Theta_2 + K_3\Theta_3 + K_{13}(\Theta_1\times\Theta_3) + K_{23}(\Theta_2\times\Theta_3) - K_{21}(\Theta_2\times\Theta_1) \tag{48}$$

where

$$K_1 = 1 + \frac{11K_A - 18K_B - 9K_C}{2}$$
$$K_2 = 1 + \frac{-7K_A - 27K_B - 18K_C}{2}$$
$$K_3 = 1 + \frac{2K_A - 9K_B - 9K_C}{2}, \qquad K_{13} = X - \frac{21K_C}{4}$$
$$K_{23} = Y + \frac{3K_C}{4}, \qquad K_{21} = Y + \frac{57K_C}{4} \tag{49}$$
$$K_A = \frac{\tilde{a}_x\tilde{c}_x + \tilde{a}_y\tilde{c}_y + \tilde{a}_z\tilde{c}_z - 2\tilde{b}_x^2 - 2\tilde{b}_y^2 - 2\tilde{b}_z^2}{120}$$
$$K_B = \frac{\tilde{a}_x\tilde{b}_x + \tilde{a}_y\tilde{b}_y + \tilde{a}_z\tilde{b}_z}{60}, \qquad K_C = \frac{\tilde{a}_x^2 + \tilde{a}_y^2 + \tilde{a}_z^2}{120}$$

The vectors $\tilde{\boldsymbol{a}}$, $\tilde{\boldsymbol{b}}$, and $\tilde{\boldsymbol{c}}$ are related with the vectors $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{c}}$ [which are determined by the formulas (14)] by the following expressions:

$$\tilde{\boldsymbol{a}} = \hat{\boldsymbol{a}}h, \qquad \tilde{\boldsymbol{b}} = \hat{\boldsymbol{b}}h^2, \qquad \tilde{\boldsymbol{c}} = \hat{\boldsymbol{c}}h^3 \tag{50}$$

Note that we did not pursue the goal to derive the exact algorithm with a minimum computational load. We have chosen the algorithm in the form of Eq. (48) because it was similar to the traditional Miller algorithm (15) and (16) and thus was suitable for an additional simulation we were going to perform. The expressions for the coefficients of Eq. (49) were obtained by appending Eq. (47) in view of Eq. (46) directly to Eq. (15) and transporting it to the form of corrections for the coefficients of the original Miller algorithm based on formulas (14).

Table 2 Performance of the nontruncated algorithm

m	0.1	0.3	0.5	0.7
r	6.06	5.89	6.13	5.99

The simulation results for different values of m are presented in Table 2. The results scarcely changed when we specified the amplitudes a_x, a_y, and a_z to be unequal ($a_x = 1.1$ deg, $a_y = 0.9$ deg, and $a_z = 1.0$ deg).

Consequently, Miller's optimization takes place for the generalized coning with the arbitrary values of m when the third term in the rotation vector differential equation is taken into account. And our procedure for optimizing attitude algorithms is quite correct.

The problem is only that the contribution of the third term of Eq. (3) should be closely studied for each type of motion being investigated. It is clearly more important for larger coning angles. Such is indeed the case for the generalized coning with large values of m, for the reason that the coefficient at the fundamental harmonic for elliptic function sn in description (36) becomes greater than 1 as m grows. The authors of Ref. 5 were quite right that the effect of generalized coning on algorithm accuracy is worse than for pure coning. But this is caused by the significant increasing of the third term's contribution, not by any changes in the conditions of optimization.

The practical significance of the obtained results is as follows: when there is no possibility to use the exact algorithm (with the third term involved), it would not be appropriate to use an algorithm with the basic computational error less than this one caused by neglecting the third term, i.e., when $\Delta\,\boldsymbol{\Phi}_1 < \Delta\,\boldsymbol{\Phi}_2$.

In concluding this section, we will turn our attention to the problem of the correct specification of the angular rate's peak values for the generalized conic motion. Note that the simulation we held for the generalized coning in the form of Eq. (33) gave the same negative results as those that were obtained in Ref. 5 but now with the nontruncated algorithm of Eq. (48). To investigate this phenomenon, we held the simulations for the classical coning but with equal peak values ($a\omega_0$) for the angular rate. In this case, too, the Miller optimization fails: the exponent r is equal to 4 with the computational drift along both x and z axes. Note that the case under investigation can be referred to the general case of regular precession, when a rigid body, executing a conic motion, additionally rotates with a constant angular rate along the axis, which traces out a conic trajectory. A study of the algorithm's performance under such an angular input should be a subject of a special study.

VII. Other Descriptions for the Generalized Coning

The description for the generalized coning presented in Sec. IV is based on the mathematically formal extension of the classical coning when the trigonometrical functions transform to the appropriate Jacobian elliptic functions.[5] However, another approach to the generalization of the conic motion, which is kinematically more correct, can be suggested.

It is well known that the classical coning is a motion characterized by a small rotation angle about the axis that rotates with the constant rate $\dot{\theta} = \omega_0$ about the X axis of the reference frame $OXYZ$ (see Fig. 1). The orientation of the instantaneous rotation axis unit vector $\boldsymbol{e}$ is defined in the frame $OXYZ$ by the angle

$$\theta(t) = \omega_0 t \tag{51}$$

An extension of the classical coning when the essence of this motion is valid but with the arbitrary angular rate $\dot{\theta}$ seems to be reasonable. In this case, the rotation vector's components along the reference frame axes can be written as

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 \\ a\cos(\theta) \\ a\sin(\theta) \end{bmatrix} \tag{52}$$

and the quaternion $\boldsymbol{Q}(t)$ is

$$\boldsymbol{Q}(t) = \begin{bmatrix} \cos(a/2) \\ 0 \\ \sin(a/2)\cos(\theta) \\ \sin(a/2)\sin(\theta) \end{bmatrix} \tag{53}$$

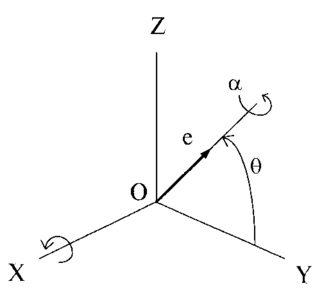

Fig. 1 Kinematics of conic motion.

Then the body frame angular rate components are obtained as

$$\boldsymbol{\omega} = \begin{bmatrix} -2\dot{\theta}\sin^2(a/2) \\ -\dot{\theta}\sin(a)\sin(\theta) \\ \dot{\theta}\sin(a)\cos(\theta) \end{bmatrix} \tag{54}$$

It is easy to see that, with expression (51), the expressions (52–54) take the conventional form of Eq. (25) of the classical coning. The property of vectors $\boldsymbol{\Phi}$ and ω being orthogonal holds true for the generalized coning as well. To derive the finite expressions for the vector ω components, it is necessary to assign a particular model for $\theta(t)$. In connection with this, the following two approaches can be considered.

1) Let the component ω_x be given according to the model (35), i.e.,

$$\omega_x = -2\omega_0 \sin^2(a/2)\mathrm{dn}(\omega_0 t, m) \tag{55}$$

We can obtain the expressions for the other two components, ω_y and ω_z, by taking into consideration the representation (54) and the known properties of the Jacobian elliptic functions. A comparison of Eq. (55) with Eq. (54) gives

$$\dot{\theta} = \omega_0 \mathrm{dn}(\omega_0 t, m) \tag{56}$$

The relation (56) can be written with the known relationship between Jacobian elliptic functions as follows:

$$\frac{\mathrm{d}\theta}{\sqrt{1 - m\,\mathrm{sn}^2(\omega_0 t, m)}} = \omega_0\,\mathrm{d}t \tag{57}$$

from which

$$\int_0^t \frac{\mathrm{d}\theta}{\sqrt{1 - m\,\mathrm{sn}^2(\omega_0 t, m)}} = \omega_0 t \tag{58}$$

It is easy to see that Eq. (58) is an elliptic integral; hence the following relations are true:

$$\sin\theta = \mathrm{sn}(\omega_0 t, m), \qquad \cos\theta = \mathrm{cn}(\omega_0 t, m) \tag{59}$$

Taking into account Eq. (56), expressions (54) can be written as

$$\boldsymbol{\omega} = \begin{bmatrix} -2\omega_0 \sin^2(a/2)\mathrm{dn}(\omega_0 t, m) \\ -\omega_0 \sin(a)\mathrm{dn}(\omega_0 t, m)\mathrm{sn}(\omega_0 t, m) \\ \omega_0 \sin(a)\mathrm{dn}(\omega_0 t, m)\mathrm{cn}(\omega_0 t, m) \end{bmatrix} \tag{60}$$

2) Let the angle $\theta(t)$ satisfy the differential equation, which describes the motion of a mathematical pendulum in the vertical plane, written as[7]

$$l\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -g\sin\theta \tag{61}$$

where l is a pendulum length and g is the acceleration of gravity. As shown in Ref. 7, the solution of Eq. (61) with the zero initial condition $\theta(0)$ can be written via Jacobian elliptic functions as follows:

$$\dot{\theta} = \dot{\theta}_0 \mathrm{dn}(u, m), \qquad \cos(\theta/2) = \mathrm{cn}(u, m) \qquad \sin(\theta/2) = \mathrm{sn}(u, m) \tag{62}$$

where $\mu = \chi_0 t/2$, $m = 2\nu^2/\chi_0$, $\nu^2 = g/l$, and $\dot{\theta}_0$ is an initial condition for the angular rate $\dot{\theta}$. Taking into account Eq. (62), we obtain

$$\omega = \begin{Bmatrix} -2\dot{\theta}_0 \sin^2(a/2)\mathrm{dn}(u, m) \\ -2\dot{\theta}_0 \sin(a)\mathrm{dn}(u, m)\mathrm{sn}(u, m)\mathrm{cn}(u, m) \\ \dot{\theta}_0 \sin(a)\mathrm{dn}(u, m)[\mathrm{cn}^2(u, m) - \mathrm{sn}^2(u, m)] \end{Bmatrix} \tag{63}$$

This model is kinematically correct though the validity of it certainly must be closely examined before usage.

The expressions (60) and (63) are the other representation for the generalized coning, which hold the kinematic essence of conic motion, because it manifests itself in relations (52–54). Note that for either of these two descriptions, the attitude algorithm can be optimized with the procedure suggested earlier. It can be shown that in each of these two cases, the optimized algorithm's coefficients are the functions of the parameter m. Considering that the alternative descriptions for the generalized coning are presented in this section mainly to illustrate the approach to the extension of the pure coning as such, the concrete expressions for optimized coefficients will not be given here.

VIII. Statistical Refinement of the Procedure

One of the basic points of Miller's procedure is the determination of the ω polynomial model's coefficients—$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, etc.—via gyro outputs Θ_i. This problem in its initial setting up is solved purely in a deterministic way. That's why in the general case the number of ω model's coefficients n_ω to be determined is exactly equal to the number of gyro samples n_θ, which are available over the iteration interval h.

Meanwhile, this problem can be set up in a statistical sense when the presence of an additive noise in the gyro outputs is taken into account. It is obvious that such a formulation of the problem is more appropriate especially due to the quantization errors. The accuracy of estimation of the vector ω model's coefficients can be improved by an increase of gyro samples that are invoked, i.e., when $n_\theta > n_\omega$. This situation can be realized either by the increase of the sampling rate, leaving the quaternion updating rate unchanged, or by the appropriate increase of the updating interval with no change of the sampling rate. In both cases, the attenuation of the measurement noise is the same. As to the algorithm's computational error (of the legitimate signal), it will remain unchanged in the first case and will increase in the second case due to the increase of the updating interval. The second variant is more common because it is often impossible to increase the sample rate. This presents a conflict. From the viewpoint of the effective noise smoothing, it is desirable to increase the iteration interval (recall that it is an interval for evaluation of ω model's coefficients). But from a viewpoint of the algorithm's computational drift value, it is desirable to decrease the iteration interval (for the fixed ω model's form). By this means, in every concrete case, the choice of the attitude algorithm (the choice of the smoothing interval's magnitude and the order of the polynomial for vector ω approximation) is a result of the compromise between these two factors.

Now let us touch on some practical aspects of how to generate a statistically optimal estimate of the ω model's coefficients in the case when $n_\theta > n_\omega$. We assume it would be convenient to use the least-squares method based on Chebyshev orthogonal polynomials.[9] To illustrate the procedure for deriving a statistically optimal algorithm, let us consider the following example.

Let the polynomial model for gyro output $\boldsymbol{\Theta}(t)$ over the updating interval $(T, T+h)$ be a quadratic one, i.e.,

$$\boldsymbol{\Theta}(T+\upsilon) = \boldsymbol{a}\upsilon + \boldsymbol{b}\upsilon^2 + \boldsymbol{c}\upsilon^3 \tag{64}$$

and four equally spaced gyro samples (Θ_1, Θ_2, Θ_3, Θ_4) are available over this interval. Hence we have the following five measurements of $\boldsymbol{\Theta}(t)$:

$$\begin{gathered} \boldsymbol{\Theta}(T) = 0, \qquad \boldsymbol{\Theta}[T+(h/4)] = \boldsymbol{\Theta}_1 \\ \boldsymbol{\Theta}[T+(h/2)] = \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 \\ \boldsymbol{\Theta}[T+(3h/4)] = \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3 \\ \boldsymbol{\Theta}(T+h) = \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3 + \boldsymbol{\Theta}_4 \end{gathered} \tag{65}$$

Recall that according to the deterministic approach three gyro samples are enough to evaluate the coefficients of model (64). The term $\Theta(t)$ can be presented via Chebyshev polynomials $\chi_i(t)$ in the form

$$\boldsymbol{\Theta}(T+t) = \boldsymbol{C}_0\chi_0(\upsilon) + \boldsymbol{C}_1\chi_1(\upsilon) + \boldsymbol{C}_2\chi_2(\upsilon) + \boldsymbol{C}_3\chi_3(\upsilon) \tag{66}$$

where $\boldsymbol{C}_0$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, and $\boldsymbol{C}_3$ are the desired coefficients.

For $\boldsymbol{C}_i$ and $\chi_i(\upsilon)$, one can obtain[9]

$$\begin{aligned} \boldsymbol{C}_0 &= \frac{4\boldsymbol{\Theta}_1 + 3\boldsymbol{\Theta}_2 + 2\boldsymbol{\Theta}_3 + \boldsymbol{\Theta}_4}{5} \\ \boldsymbol{C}_1 &= \frac{2(2\boldsymbol{\Theta}_1 + 3\boldsymbol{\Theta}_2 + 3\boldsymbol{\Theta}_3 + 2\boldsymbol{\Theta}_4)}{5h} \\ \boldsymbol{C}_2 &= \frac{8(-2\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3 + 2\boldsymbol{\Theta}_4)}{7h^2} \\ \boldsymbol{C}_3 &= \frac{16(\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2 - \boldsymbol{\Theta}_3 + \boldsymbol{\Theta}_4)}{3h^2} \end{aligned} \tag{67}$$

$$\begin{gathered} \chi_0(\upsilon) = 1, \qquad \chi_1(\upsilon) = \upsilon - (h/2) \\ \chi_2(\upsilon) = \upsilon^2 - h\upsilon + (h^2/8) \\ \chi_3(\upsilon) = \upsilon^3 - \tfrac{3}{2}h\upsilon^2 + \tfrac{43}{80}h^2\upsilon - \tfrac{3}{160}h^3 \end{gathered} \tag{68}$$

Going from the model (66) to the model (64) for $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ estimates of the coefficients, the following can be written:

$$\begin{aligned} \hat{\boldsymbol{a}} &= (1/28h)(125\boldsymbol{\Theta}_1 - 11\boldsymbol{\Theta}_2 - 59\boldsymbol{\Theta}_3 + 29\boldsymbol{\Theta}_4) \\ \hat{\boldsymbol{b}} &= (1/14h^2)(-9\boldsymbol{\Theta}_1 + 6\boldsymbol{\Theta}_2 + 8\boldsymbol{\Theta}_3 - 5\boldsymbol{\Theta}_4) \\ \hat{\boldsymbol{c}} &= (9/4h^3)(\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2 - \boldsymbol{\Theta}_3 + \boldsymbol{\Theta}_4) \end{aligned} \tag{69}$$

The algorithm for rotation vector computation finally is

$$\begin{aligned} \boldsymbol{\Phi}(T+h) &= \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3 + \boldsymbol{\Theta}_4 + \alpha_1(\boldsymbol{\Theta}_1 \times \boldsymbol{\Theta}_2) \\ &+ \alpha_2(\boldsymbol{\Theta}_1 \times \boldsymbol{\Theta}_3) + \alpha_3(\boldsymbol{\Theta}_1 \times \boldsymbol{\Theta}_4) + \alpha_4(\boldsymbol{\Theta}_2 \times \boldsymbol{\Theta}_3) \\ &+ \alpha_5(\boldsymbol{\Theta}_2 \times \boldsymbol{\Theta}_4) + \alpha_6(\boldsymbol{\Theta}_3 \times \boldsymbol{\Theta}_4) \end{aligned} \tag{70}$$

where

$$\alpha_1 = \alpha_6 = \tfrac{52}{105}, \qquad \alpha_2 = \alpha_5 = \tfrac{212}{315}, \qquad \alpha_3 = \tfrac{112}{315}, \qquad \alpha_4 = \tfrac{184}{315}$$

The algorithm with four gyro samples over the iteration interval was derived in Ref. 3. This algorithm was based on the cubic model for vector ω and was obtained with the conventional deterministic approach. The algorithm of Eq. (70) has the same form but it has other coefficients, for it possesses the property of measurement noise smoothing.

We have no intention of examining the efficiency of such an approach in this paper. It may be the subject of a future publication.

IX. Conclusion

A new procedure for deriving optimized strapdown attitude algorithms was presented and justified. It is based on Miller's approach and allows optimization of the algorithm for the case when the analytical expressions for the vehicle angular rate's components are known. The distinctive feature of this procedure is that the expression for the algorithm's computational error was obtained in a general form directly for the rotation vector without deriving the expression for the quaternion (or rotation vector) value generated by the algorithm and moreover without the necessity to know the expressions for the true values of these parameters. The condition for optimizing the algorithm is the condition of compensating the computational error, and it was obtained using the analytical relationship between angular rate derivatives. The procedure was tested on the classical coning motion for Miller's algorithm with three gyro samples over the iteration interval, and the known optimized coefficients were obtained.

With the suggested procedure, Miller's algorithm was optimized for the generalized coning when the vehicle angular rate components were the Jacobian elliptic functions as described in Ref. 5. It was shown (analytically and by simulation) that the coefficients for the classical coning hold true for the generalized coning as well but only when the following two conditions are met.

1) The angular rate component's peak values must be properly specified as typical for the conic motion.

2) The third term in the rotation vector differential equation has to be taken into account.

A conclusion was made based on the study of the third term's contribution that, when the nontruncated algorithm cannot be applied, it would not be appropriate to try to design an algorithm with a computational error less than the error obtained when the third term is neglected.

Retaining the kinematic essence of conic motion, an extension of such a motion was obtained in a general form. On the basis of this representation, two alternative examples of the generalized coning were derived.

A statistical refinement of the procedure that allows us to optimize the algorithm from the viewpoint of the measurement noise smoothing was suggested. An example of deriving the algorithm with smoothing properties based on the least-squares method with the Chebyshev orthogonal polynomial is presented. It is stated that, in every concrete case, the choice of the algorithm is a result of the compromise between the noise smoothing and the accuracy of the vector ω model's determination.

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